

Two-fold branched covers

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January 1, 2013

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Introduction

Every closed, orientable 3-manifold is a 3-fold branched cover of S^3 . This was independently shown by Hilden, Hirsh and Montesinos [15, 16, 22]. Many three dimensional manifolds are two-fold branched covers of the three dimensional sphere, S^3 . However, there are 3-manifolds that are not. For example, R. H. Fox showed that the 3-torus is not a 2-fold branched cover of S^3 [11]. In this paper, we explain why many 3-manifolds are 2-fold branched covers of S^3 , show that there are integral homology spheres that are not 2-fold branched covers of S^3 , but are 2-fold branched covers of some manifold, and there are integral homology spheres that do not 2-fold cover any 3-manifold. When a 3-manifold is obtained as surgery on a hyperbolic knot, the manifolds that are 2-fold branched covered by it can be understood via the knot, and except for a finite number of possible exceptions there is a bijection between the branched covering projections of the knot complement and the branched covering projections of the manifold obtained by surgery. We also show that every 3-manifold branched virtually fibers i.e. has a finite *branched* cover that fibers over the circle, S^1 .

I would like to thank B. Owens for asking me if there is a rational homology sphere that is not a 2-fold branched cover.

First Examples

It is natural to consider 3-manifolds that are obtained as surgery on a knot. For knots with fewer than 11 crossings, it turns out that most of these 2-fold

branch cover S^3 . We will now discuss several qualitatively different examples of 2-fold branched covers and review the standard constructions from low-dimensional topology that we use.

Example 1

A good example to consider first is $2/3$ Dehn surgery on the 5_2 knot displayed on the left in figure 1. The knot and the framing curve are clearly set-wise invariant under a 2-fold rotation of S^3 . This rotation will surger to the deck transformation of a 2-fold branched covering from the sugrered manifold to S^3 .

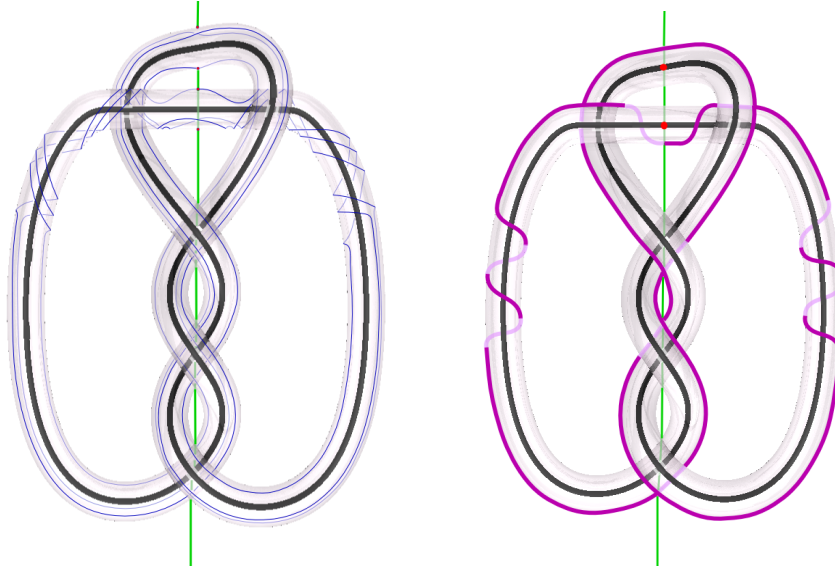


Figure 1: The $2/3$ slope on the 5_2 knot and the longitude

Recall that Dehn surgery is the result of removing an open solid torus and gluing in a closed solid torus:

$$M_K(p/q) := \left(S^3 - \mathring{N}(K) \right) \cup_{T^2} (D^2 \times S^1).$$

The open solid torus is a tubular neighborhood of a knot, $\mathring{N}(K)$. The complement of an open tubular neighborhood of a knot is known as the exterior of the knot. For oriented knots in S^3 generators of the first homology

of the boundary of the tubular neighborhood can be chosen in a canonical way. The longitude, λ , is the class of an oriented section of the normal bundle that bounds in the complement of the knot. The longitude of the 5_2 knot is displayed on the right in figure 1. The meridian, μ , is the class of the boundary of a fiber of the normal bundle oriented so that its linking number with the knot is positive. When the surgery slope, $1 \times \partial D^2$, represents the homology class $p\mu + q\lambda$, the surgery coefficient is $p/q \in \mathbb{Q} \cup \{\infty\}$.

The $2/3$ -curve on the boundary of a tubular neighborhood of the 5_2 knot has a representative that is set-wise fixed by the 2-fold rotation. From the picture, it is not clear that every curve class has a representative that is set-wise fixed, and it is not clear what the quotient manifold is. We will see that every slope is represented by a set-wise fixed curve and the the quotient is just S^3 and see that this is a very general phenomena.

We first verify that the original 2-fold rotation induces an involution on the surgered manifold. This rotation restricts to an elliptic involution of the torus. This is the map $\tau : S^1 \times S^1 \rightarrow S^1 \times S^1$ given by $\tau(z, w) := (\bar{z}, \bar{w})$ using complex coordinates. Here we identify the torus with the boundary of the exterior of the knot so that $(z, 1)$ is a meridian and $(1, w)$ is a longitude.

In the surgery process we adjoin a solid torus $D^2 \times S^1$ to the boundary of the exterior of the knot. Viewing $D^2 \times S^1 := \{(z, w) \in \mathbb{C}^2 \mid |z| \leq 1, |w| = 1\}$, the identification $j : S^1 \times S^1 \rightarrow D^2 \times S^1$ can be specified by a matrix in $\text{SL}_2\mathbb{Z}$, say A , such that $j(\exp(2\pi i v)) = \exp(2\pi i A^{-1}v)$ with $v = (x, y)$ and the exponential acting component wise. This is because every orientation preserving diffeomorphism of a torus is isotopic to one of this form and changing the identification by an isotopy does not change the resulting manifold. These two facts are nicely explained in [29]. The surgery coefficients are given by $(p, q) = A(1, 0)$.

To see that the involution extends to the surgered manifold, just define it on $D^2 \times S^1$ by the same formula: $\tau(z, w) := (\bar{z}, \bar{w})$ and notice that $j((\bar{z}, \bar{w})) = \overline{j(z, w)}$. Thus the original involution induces an involution on any manifold obtained by Dehn surgery on the knot. The fixed point locus in the knot exterior is a pair of intervals as is the fixed point set in the solid torus. Thus all of these manifolds are 2-fold branched covers of some 3-manifold.

An alternative proof that the involution extends will show that each of these manifolds is a 2-fold branched cover of S^3 . Figure 2 shows that the quotient of a solid torus by the elliptic involution is a closed 3-ball. It follows that the quotient of Dehn surgery on a knot by an involution that acts as the elliptic involution on the boundary of the tubular neighborhood is obtained



Figure 2: Quotient of $D^2 \times S^1$ by the involution

by removing a 3-ball from the 3-sphere and gluing in a closed 3-ball via some homeomorphism. Since every homeomorphism of S^2 extends across the 3-ball the quotient of the original manifold must be S^3 .

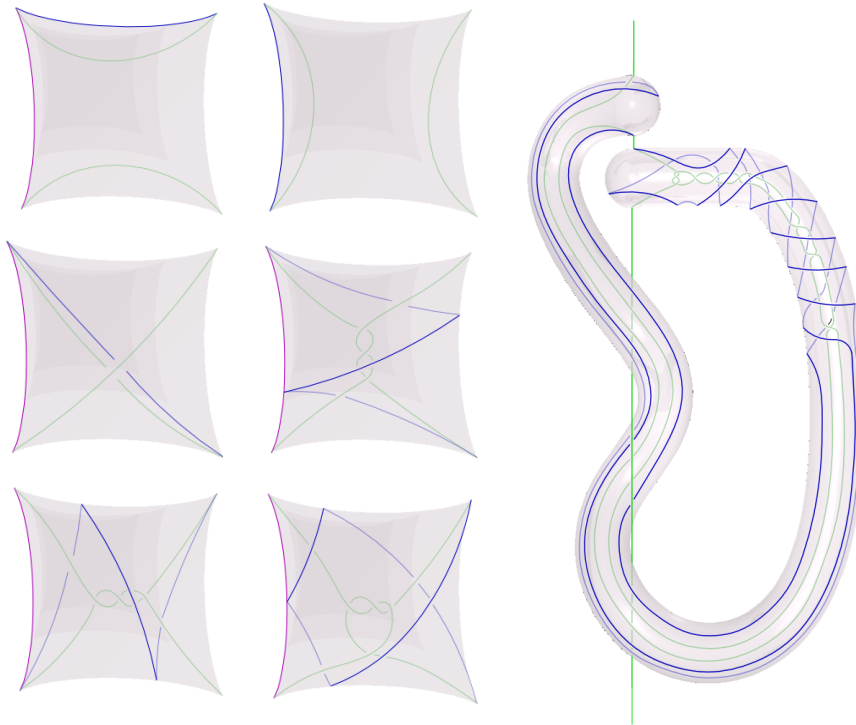


Figure 3: Tangling the branched set

Every matrix in $\mathrm{SL}_2\mathbb{Z}$ is a product of copies of $S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Since these both commute with $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, they also act on the quotient of T^2 by the elliptic involution. This quotient is an orbifold known as the pillow case or 2222-orbifold. The underlying topological space is S^2 and the singular locus consists of four cone points of order 2. The apparent left edge of the pillow case lifts to a longitude of the original knot. The apparent top edge of the pillow case lifts to a meridian.

The top left portion of figure 3 displays the pillow case before any homeomorphism is applied. We can label each pillow case in this figure by the corresponding surgery slope, so this first one would be labeled by $\infty := 1/0$. The matrix S acts as a 90 degree rotation of the pillow case, and this clearly extends to the pillow. The matrix T acts as a twist that interchanges the two corners on the right hand side. Figure 3 shows the the pillow as the sequence of homeomorphisms S, T, T^2, S, T is applied. We could label the pillows 0, 1, 3, $-1/3$, and $2/3$. In each pillow the image of the top edge is colored blue, and the branch locus is colored green. The apparent left edge is consistently colored purple. If the resulting pillow is glued into the quotient of a knot exterior so that the apparent left edge maps to the image of the longitude and the apparent top edge maps to the meridian the resulting configuration will be the image of the surgery slope and the surgered solid torus together with the branch locus in the quotient manifold. Figure 3 shows the result of gluing the $2/3$ -pillow configuration into the image of the exterior of the 5_2 knot. Lifting the blue curve produces the invariant representative of the $2/3$ -slope that was displayed in figure 1.

It should now be clear that this procedure could be followed with any surgery slope and any knot admitting an involution that restricts to the elliptic involution on the boundary of the tubular neighborhood. This procedure also produces an explicit description of the branch locus in the quotient. This completes the alternate proof that the quotient of this type of involution is S^3 .

Example 2

Figure 4 displays a projection of the 10_{98} knot. This projection is invariant under a $1/2$ rotation about the axis coming out of the page through the green dot. The main difference between this example and the previous example is

that in this case the axis of the rotation does not intersect the knot. This means that the quotient of the pair (S^3, K) is (S^3, K') where K' is a copy of S^1 embedded in S^3 . In this case it is knotted. Once again we can verify that

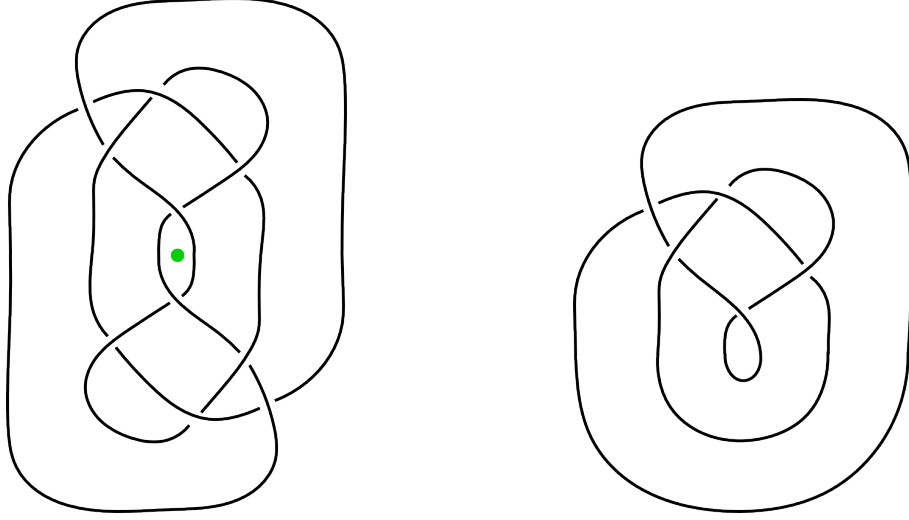


Figure 4: The 10_{98} knot and its quotient

this rotation induces an involution on any manifold obtained by surgery on the 10_{98} knot.

To begin, notice that the 2-fold branched cover of a generic Seifert surface in the quotient is a Seifert surface for the original knot. This means that the $1/2$ rotation fixes a longitude of the knot set wise. The same rotation does not fix a meridian. It follows that the boundary may be identified with a torus so that the rotation is given by $\tau(z, w) = (z, -w)$. For p/q surgery on the knot, choose integers r and s so that $ps - qr = 1$ and define a deck transformation on $D^2 \times S^1$ by $\tau(z, w) = ((-1)^r z, (-1)^p w)$. It is not difficult to check that this deck transformation is compatible with the identification of the boundary of $D^2 \times S^1$ with the boundary of the tubular neighborhood of the knot.

A bit more information can be seen. When p is even, r must be odd and $\{0\} \times S^1$ will be an extra component in the branch locus which will have two components. When p is odd, the axis in the knot exterior will be the only component of the branch locus. Since a meridian in the quotient lifts to two meridians, and a longitude in the quotient lifts to just one longitude, the surgery coefficient in the quotient will be exactly one half of the surgery

coefficient on the original knot.

Restrictions on covering projections

We have seen two fairly general constructions of 2-fold branched covers. In this short section we will explain how we can often identify all 2-fold branched quotients of a manifold. The key idea is that a 2-fold branched cover is determined by its unique non-trivial deck transformation and that it suffices to consider isometries when looking for such deck transformations. State-of-the-art 3-manifold theory is sufficiently advanced that this can be easily accomplished by quoting from the many deep theorems in the area. These include the orbifold theorem, the geometrization theorem and Thurston's hyperbolization theorem.

See [1] for the modern definition of an orbifold. An orbifold is essentially a special type of groupoid, and a groupoid is just a category such that every morphism has an inverse. In the case of a 2-fold branched cover $p : M \rightarrow N$ we let the objects equal the points in M and let the arrows be the pairs (x, τ) where $x \in M$ and τ is a deck transformation, i.e. a self homeomorphism of M such that $p \circ \tau = p$. The space underlying this orbifold is just the set of equivalence classes of objects under the relation that identifies objects connected by an arrow. This is just a fancy way to add groups to the points, namely the group of self morphisms of each point.

The orbifold geometrization theorem, [6, 5] is stated below for the reader's convenience.

Theorem 1 (Thurston; Cooper, Hodgson, Kerckhoff; Boileau, Leeb, Porti). *If \mathcal{N} is a compact, connected, orientable, irreducible, atoroidal, 3-orbifold with non-empty singular locus, then \mathcal{N} is geometric.*

We will also use Thurston's hyperbolization theorem and the full geometrization theorem to establish the following result that provides effective restrictions on the possible quotients of 2-fold covering projections.

Theorem 2. *If M^3 is a compact, connected, orientable, hyperbolic 3-manifold and $p : M^3 \rightarrow N^3$ is a 2-fold branched cover, then there is a hyperbolic metric on M for which every deck transformation is an isometry.*

Proof. We first notice that N is the underlying space of an orbifold, \mathcal{N} . We first consider the case when the branch set is non-empty and apply the

orbifold theorem. Clearly, \mathcal{N} is compact and connected. Let τ be the non-trivial deck transformation, and let g be the metric on M . The averaged metric $\frac{1}{2}(g + \tau^*g)$ is certainly τ -invariant (but it might no longer be hyperbolic). If \mathcal{N} was non-orientable, $d\tau$ would have to be an orientation reversing isometry (with the averaged metric) of the tangent space of any fixed point, so $-I$ or a reflection. In either case, N would fail to be a manifold near the orbit of the fixed point.

A bad orbifold is one that is not covered by a manifold. The quotient \mathcal{N} cannot contain a bad orbifold because it is covered by a manifold and the lift of the supposed bad orbifold would have to lift to a manifold. A spherical 2-orbifold in the quotient would lift to a sphere in M . Since M is hyperbolic, it is irreducible so the lift will bound a 3-disk in M . The quotient of this 3-disk will be a discal orbifold in \mathcal{N} bounding the given spherical orbifold. It follows that \mathcal{N} is irreducible. A similar argument (lift, find a compressing disc and take the quotient) will show that \mathcal{N} is atoroidal.

Applying the orbifold theorem tells us that there is a geometric (locally homogeneous) metric on \mathcal{N} , say h . This lifts to a geometric metric \tilde{h} on M and this must be hyperbolic as the Gromov norm of a manifold with any of the other seven geometric structures must be zero and the Gromov norm of a hyperbolic manifold is proportional to its volume, [3, 6].

If the branch set is empty we can apply Perelman's proof of the geometrization theorem to obtain the same result, [26, 28, 27]. This theorem is discussed in [19, 9, 24, 4]. Some people may worry about the case when the quotient N is non-orientable. The geometrization theorem is certainly true in this case because such an N will be Haken, so the original work of Thurston applies, [31]. See also the book [18]. (It is certainly appropriate to quote Thurston here as he is the one who originally pointed out the power of applying geometric techniques to the study of 3-manifolds.) The fact that N is Haken when it is non-orientable may be found in Hempel's book, [13].

The idea is for the solution to our problem when the fixed point set is empty is the same as when the fixed point set is non-empty. One must prove that the quotient is hyperbolic, in this case it suffices to show that it is irreducible, atoroidal, with infinite fundamental group. All of these properties follow because the 2-fold cover M has the same properties. \square

The previous theorem states that there is a hyperbolic metric on the manifold M for which the deck transformations are isometries. One may worry that there are different hyperbolic metrics that need to be considered.

However, it is known that there is a unique (up to isometry) hyperbolic metric on any closed hyperbolic 3-manifold. This is Mostow's rigidity theorem, [25].

Given the previous theorem it makes sense to see what is known about the isometries of hyperbolic 3-manifolds. Once again, the known results far surpass what is needed for our present problem. One would like to say that any smooth automorphism of a manifold induces an isomorphism of the fundamental group. The problem with this is that such an automorphism may very well move the base point. If the manifold is connected one may connect the base point to its image with a path and thereby get an isomorphism of the fundamental group. The problem with this is that changing the choice of path will change the isomorphism by conjugation. The group of isomorphisms modulo conjugation (inner automorphism) is known as the outer automorphism group. It is denoted by $\text{Out}(\pi_1(M))$. Since homotopic maps induce the same map at the level of the fundamental group, there is a well defined map

$$\text{Diff}(M)/\text{Diff}(M)_0 \rightarrow \text{Out}(\pi_1(M)).$$

Here $\text{Diff}(M)_0$ denotes the path component of the identity. Of course any isometry is also a diffeomorphism, so one may map the isometries to the diffeomorphisms and the mapping class group $\text{Diff}(M)/\text{Diff}(M)_0$. It is a consequence of Mostow's rigidity theorem that the maps relating the three groups $\text{Diff}(M)/\text{Diff}(M)_0$, $\text{Out}(\pi_1(M))$, and $\text{Isom}(M)$ are all isomorphisms for finite volume hyperbolic 3-manifolds, [3, 25].

We remark that the following deep theorem due to D. Gabai gives even more information about the diffeomorphisms of a hyperbolic 3-manifold.

Theorem 3 (Gabai). *If M is a closed, hyperbolic 3-manifold, then the inclusion of the isometry group $\text{Isom}(M)$ into the diffeomorphism group $\text{Diff}(M)$ is a homotopy equivalence.*

Hodgson and Weeks developed an algorithm that can compute the isometry groups of many hyperbolic 3-manifolds, [17]. Basically, the isometries of the complement of any hyperbolic link in a closed, hyperbolic 3-manifold that take meridians to meridians induce isometries of the closed 3-manifold. This gives a lower bound on the size of the isometry group. The total number of isometries of a closed, hyperbolic 3-manifold is bounded above by the product of the order of the group of isometries fixing a given closed geodesic and the number of geodesics having the same complex length. This algorithm is implemented in the SnapPy program.

One should not worry about the level of rigor of the computer calculations. Symmetries of knots can be analyzed with topological tools. The extensive theory of characteristic splittings of knots as developed by Bonahon and Siebenmann provides a powerful framework to address such questions [7]. Kodama and Sakuma used topological arguments to analyze the symmetry groups of all prime knots with fewer than eleven crossings, [20].

As a first application of the fact that we can restrict our attention to isometries and it is possible to compute isometries, we computed the symmetry group of $S^3_{1098}(1)$ to be \mathbb{Z}_2 . Thus there is only one element of order two, and it is the involution generated by a $1/2$ rotation about the axis in figure 4. Thus it admits a double branched covering projection to $S^3_{31}(1/2)$, but does not double branch cover any other 3-manifold (including S^3).

In addition to asking about the existence of 2-fold branched cover quotients of a 3-manifold, one may ask questions about the complexity of such quotients. One way to measure the complexity of such a quotient would be via the number of components of the branch set. The following result is a special case of a proposition derived with Smith theory. The more general result may be found on page 376 of Bredon's book, [8].

Theorem 4. *If τ is an involution on a closed, connected, oriented 3-manifold and it acts trivially on the homology, then the number of components of the fixed point set of τ satisfies the following bound:*

$$\text{Number of components} \leq 1 + \dim_{\mathbb{Z}_2}(H^2(M; \mathbb{Z}_2)).$$

More Examples

In this section we consider additional examples representing the behavior of all relevant involutions. There are exactly two involutions of a circle up to conjugation in the homeomorphism group: reflection, and half-rotation. Similarly, up to conjugation by a homeomorphism there are exactly four involutions of S^3 . They are given by sending (x_1, x_2, x_3, x_4) to one of $(-x_1, x_2, x_3, x_4)$, or $(-x_1, -x_2, x_3, x_4)$, or $(-x_1, -x_2, -x_3, x_4)$, or $(-x_1, -x_2, -x_3, -x_4)$. This follows from the resolution of the Smith conjecture [23]. The involutions of S^3 are specified by their fixed point set. The fixed point sets of the four involutions are S^2 , S^1 , S^0 , and \emptyset respectively. Thus one can categorize knots that are setwise invariant under an involution into types according to the fixed point set in S^3 and the fixed point set in S^1 . The possibilities are:

- (S^2, S^1) Here the only possibility is the unknot. This type of involution does not lead to branched covers.
- (S^2, S^0) Here the only possibilities are the unknot and composite knots. This does not lead to branched covers either.
- (S^1, S^1) Here the only possibility is the unknot. This gives the standard 2-fold branched covering projection from S^3 to itself.
- (S^1, S^0) This is fairly common among small crossing number knots. The 5_2 knot admits such a symmetry as seen in our first example (figure 1). The knot projects to an arc in the quotient, thus this type of symmetry induces 2-fold branched covering projection from any filling of such a knot complement to S^3 .
- (S^1, \emptyset) This is also fairly common among small crossing number knots. The 10_{98} knot admits such a symmetry (figure 4). This type of symmetry always leads to a 2-fold branched covering projection. Here it is worth keeping track of whether the quotient of the knot is knotted or the unknot. This is because the quotient manifold will be a non-trivial 3-manifold when the original manifold is non-trivial and the quotient of the knot is knotted, or if the original manifold is not a homology sphere and the quotient of the knot is the unknot.
- (S^0, S^0) This may occur as seen with the 8_{17} knot (figure 6). The symmetry only extends to the trivial filling or to 0-filling. With 0-filling the resulting symmetry induces a 2-fold branched cover, to a non-orientable manifold.
- (S^0, \emptyset) This may also occur, but it will never lead to a branched cover, as the quotient would have a neighborhood homeomorphic to a cone on the projective plane.
- (\emptyset, \emptyset) This may also occur as seen with the 10_{155} knot (figure 9). Since the \mathbb{Z}_2 action is free, the branch set would be contained in the solid torus added in the filling, or the induced quotient projection will be an unramified 2-fold cover. The quotient will also be orientable, since the non-trivial deck transformation preserves orientation.

Example 3

For our paper symmetries of types (S^1, S^0) and (S^1, \emptyset) will be the most important because these are the main types that produce interesting 2-fold branched covers. It is quite common for small crossing number knots to admit both types of symmetries. The 5_2 knot admits both, as can be seen in the projection on the left of figure 5. This projection is invariant under $1/2$ rotations about the blue horizontal axis, the green vertical axis, and an axis coming out of the center of the page. The blue axis misses the knot, and the other two axes both meet the knot in two points. The image of the knot in the quotient is just the unknot.

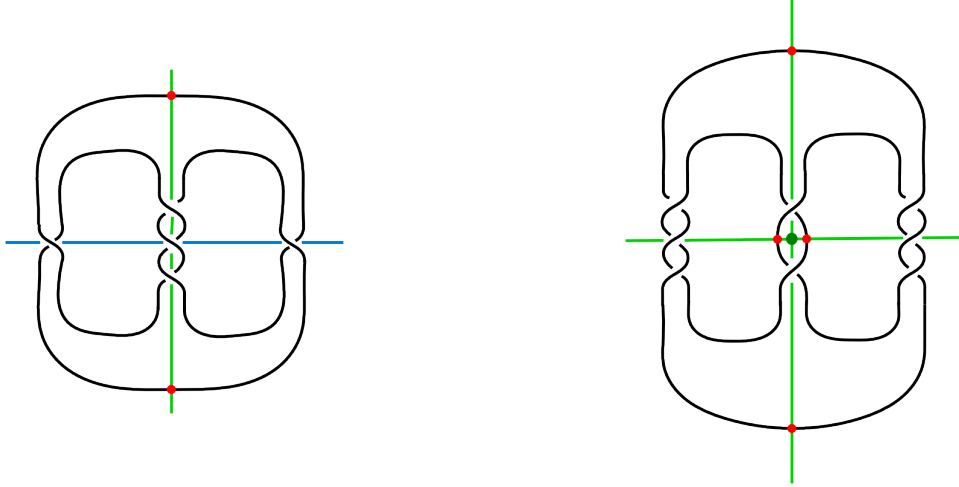


Figure 5: Full symmetry of 5_2 and 8_5

The 8_5 knot has the same symmetry group, but the image of the knot under the quotient of the rotation that doesn't fix any point on the knot is knotted (it is a trefoil). This implies that any manifold obtained by non-trivial surgery on the 8_5 knot is a 2-fold branched cover of S^3 (via deck transformations induced by rotations that fix points on the knot), and is a 2-fold branched cover over a 3-manifold that is not S^3 (via deck transformations induced by rotations that do not fix any point on the knot.) This includes the integer homology spheres obtained by $1/n$ surgery. We know that non-trivial surgeries on non-trivial knots are non-trivial manifolds by the resolution of

the property P conjecture [21].

The same argument shows that any non-homology sphere that is surgery on the 5_2 knot both 2-fold branch covers S^3 and non-simply connected manifolds. The manifold $S^3_{5_2}(1/3)$ has symmetry $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as expected from the left side of figure 5. Each of the involutions generates a double branched covering to S^3 , thus this manifold only branched double covers S^3 .

Examples 4 and 5

We have seen that many knots admit symmetries of types (S^1, S^0) and (S^1, \emptyset) , so that most surgeries on these knots 2-fold branched cover S^3 and a non-simply connected manifold. The 10_{98} knot from example 2 had only a symmetry of type (S^1, \emptyset) , thus generic surgeries on it 2-fold branched cover non-simply-connected manifolds, but do not 2-fold branched cover S^3 . In the other direction, the 8_{10} knot from figure 6 admits only a symmetry of type (S^1, S^0) . Thus generic surgeries on this knot 2-fold branched cover S^3 but no other manifold.

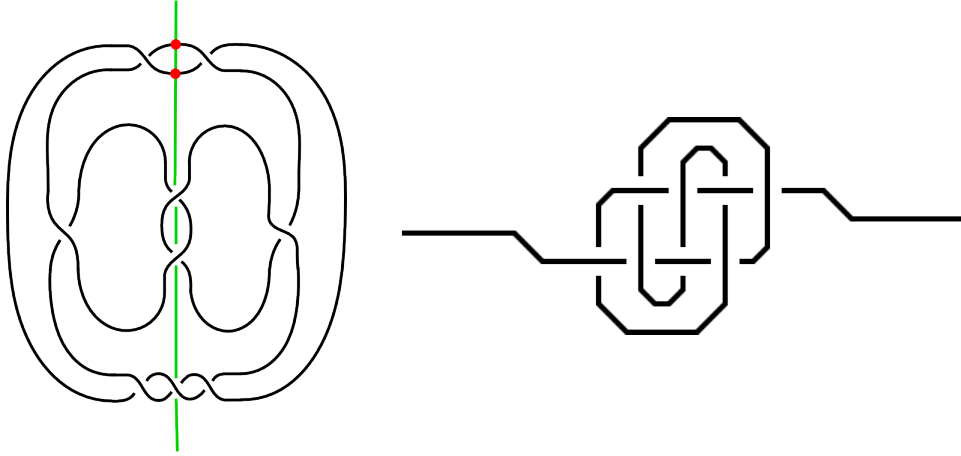


Figure 6: The 8_{10} and 8_{17} knots

The 8_{17} knot from figure 6 is an interesting example. The only symmetry it admits is of type (S^0, S^0) . This symmetry restricts to a half rotation on the meridian and to a reflection on the longitude. For this symmetry to extend the surgery slope must be taken to itself. As $p\mu + q\lambda$ is mapped

to $p\mu - q\lambda$ we see that this can only occur if $p/q = \infty$ or $p/q = 0$. For zero surgery, the symmetry extends over the $S^1 \times D^2$ via $(z, w) \mapsto (-z, \bar{w})$. This has no fixed points, so the resulting quotient projection is a covering transformation. Since the symmetry is orientation reversing, the quotient is non-orientable. Generic surgeries on this knot will have no symmetries, so the resulting manifolds will not 2-fold branched cover any manifold.

The knot in figure 10 has no symmetries, so generic surgeries on this knot do not 2-fold branched cover any 3-manifold. The smallest knot with no symmetries is the 9_{32} knot.

Example 6 and Torus Knots

The quotient of S^3 by an involution with no fixed points is $\mathbb{R}P^3$. Thus to get a knot with a symmetry of type (\emptyset, \emptyset) , one may take any homologically non-trivial knot in $\mathbb{R}P^3$ and consider its lift into S^3 . Easy examples are given by torus knots $T(p, q)$ with both p and q odd.

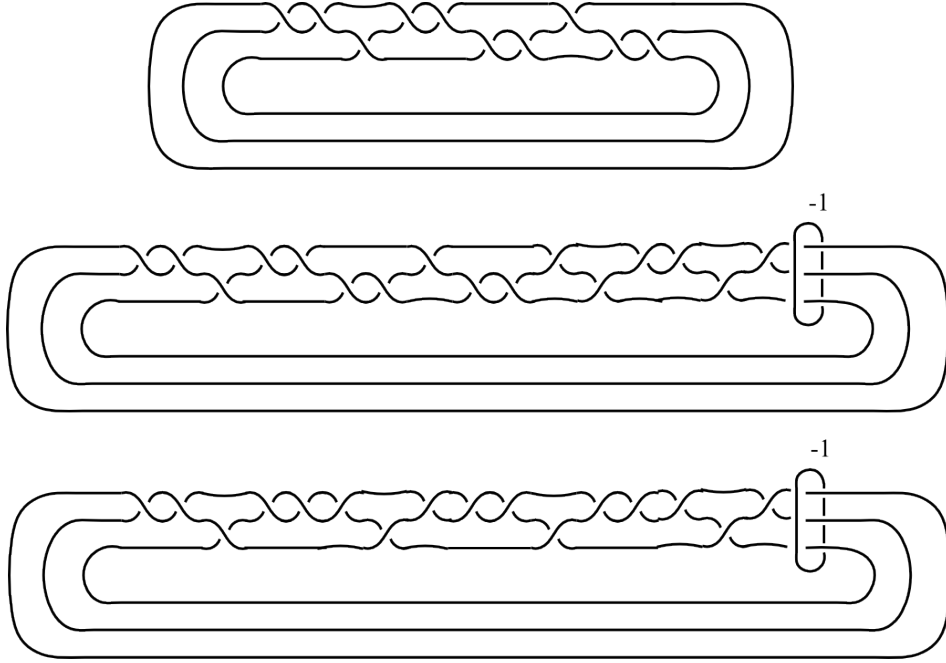


Figure 7: The $T(3, 5)$ (10_{124}) knot

Recall that a torus knot is given by (λ^p, λ^q) inside S^3 viewed as pairs of complex numbers with $\max(|z|, |w|) = 1$. The antipodal involution $(z, w) \mapsto$

$(-z, -w)$ clearly has no fixed points and preserves the torus knots $T(p, q)$ setwise for odd p and q . It also takes the meridian to the meridian and the longitude to the longitude homologically, so any surgery extends. Since this involution has no fixed points, it does not even fix a meridian of the knot setwise. A longitude of the torus knot is given by the boundary of a Seifert surface. In this case a Seifert surface is given by the radial projection of $\{(z, w) \in \mathbb{C}^2 | z^q - w^p = 1\}$. It is clear that such a surface is disjoint from its image under the antipodal involution. It follows that this rotation does not fix a longitude of the knot setwise.

The special case of $T(3, 5)$ is displayed in figure 7, along with a surgery description. The antipodal involution rotates both factors of the solid torus one-half way around. A left-handed Rolfsen twist adds the full left twist that appears on the right side of the braid in the second part of figure 7. Pushing half of this twist past half of the braid results in a representation in which the antipodal involution may be seen as a one-half rotation. The quotient is then easily recognized as a homologically non-trivial knot in $\mathbb{R}P^3$. The same procedure will work with any knot having this type of symmetry. The quotient is displayed in figure 8. This figure further simplifies the quotient to make the Seifert fibered structure apparent.

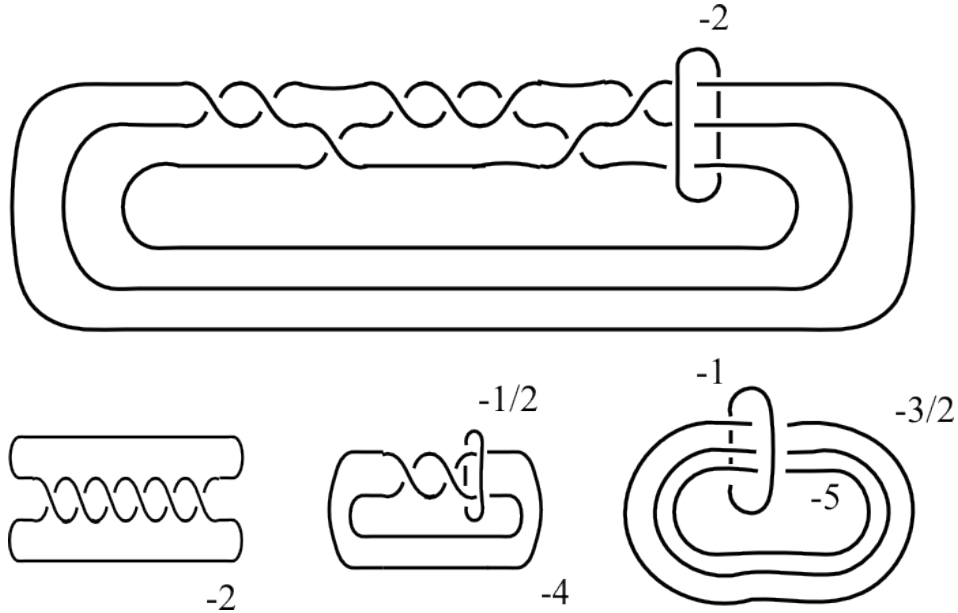


Figure 8: The quotient of the $T(3, 5)$ knot

Since neither the meridian or longitude is fixed setwise, the rotation of the boundary torus induced by the antipodal involution is given by $(x, y) \mapsto (x + \frac{1}{2}, y + \frac{1}{2})$. It follows that the indivisible homology class $r\mu + s\lambda$ on the boundary has a setwise fixed representative if and only if there is a $t \in \mathbb{R}$ such that $rt \cong st \cong 1 \pmod{1}$, and this holds if and only if r and s are both odd.

When there is no such setwise fixed representative, the involution extends as a free involution to the r/s filling, so the resulting filling is an unramified 2-fold cover of a non-trivial 3-manifold. When there is a setwise fixed representative, the core of the filling torus will be the branch locus of the involution. The same arguments may be used with any knot admitting a symmetry of type (\emptyset, \emptyset) .

When there is a setwise fixed representative, we do not need the explicit surgery descriptions to understand the quotients of the torus knots because the exterior of each of these has the structure of a Seifert fiber space given by the group action $(z, w) \cdot \lambda = (\lambda^p z, \lambda^q w)$. The orbits of $(1, 0)$ and $(0, 1)$ are singular fibers of this Seifert fibration. The Seifert invariants of these singular fibers are (p, u) and (q, v) where u and v are integers with $pu + qv = 1$. For odd (p, q) the antipodal involution preserves the fibers so the quotient will also be a Seifert fiber space. This remains the case for all Dehn fillings. The Seifert invariants of the quotient will be

$$\{0, (Oo, 0), (p, 2u), (q, 2v), (r, s)\}.$$

Unless $r = \pm 1$ the fundamental group of this manifold will surject onto the (p, q, r) triangle group. In general, the order of the first homology of this manifold is $|2r - pqs|$, and it is easy to check that this will never be one when $r = \pm 1$ and p , and q are odd and relatively prime.

While we are discussing torus knots, notice that the involution $(z, w) \mapsto (\bar{z}, \bar{w})$ preserves any torus knot setwise, thus every one has a symmetry of type (S^1, S^0) and therefore any surgery on a torus knot 2-fold branched covers S^3 . There is an important difference between torus knots and hyperbolic knots. Whereas the isometry group, mapping class group and outer automorphism group of a finite volume hyperbolic manifold are all isomorphic and all finite, this is no longer true for Seifert fiber spaces and torus knots. For example, the isometry group of the torus knots include the following 1-parameter subgroup of isomorphisms: $f_\lambda(z, w) = (\lambda^p z, \lambda^q w)$, so the type (\emptyset, \emptyset) isometries of torus knots described above are non-trivial as isometries, but are trivial elements of the mapping class group.

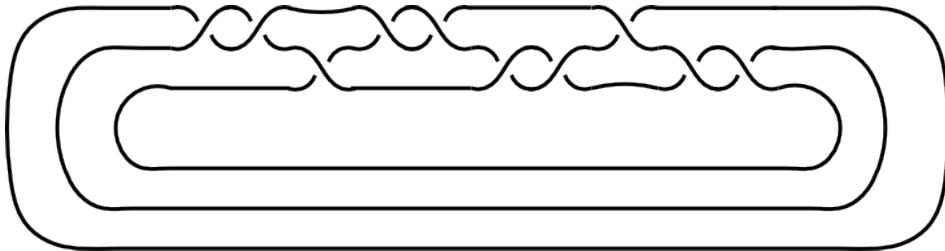


Figure 9: The 10_{155} knot

The 10_{155} knot is displayed in figure 9. It is obtained from the $T(3, 5)$ torus knot by symmetrically changing four crossings. It follows that it has the same two involutions – one of type (S^1, S^0) and one of type (\emptyset, \emptyset) . As always the (S^1, S^0) symmetry induces 2-fold branched covering projections from any Dehn filling to S^3 . Since it is a hyperbolic knot, the quotient of its exterior under the free action will be hyperbolic. It follows from property P that the induced quotients on all but at most one Dehn filling will be non-trivial 3-manifolds. Surgery descriptions of these quotients may be constructed by following the procedure used in figure 7 and figure 8.

Exceptional Symmetries

We have seen that the possible quotients of 2-fold branched covers of a 3-manifold are determined by the involutions of the manifold. Thus if one understands all symmetries of the manifold one also understands all such quotients. When the manifold is surgery on a knot these symmetries can generally be understood via symmetries of the knot. This is similar to the situation with hyperbolic structures. Thurston’s hyperbolic Dehn surgery theorem [30] states that all but a finite number of Dehn fillings of a complete hyperbolic manifold with one cusp end admit hyperbolic structures. A similar thing holds true for symmetries. For all but a finite number of Dehn fillings of a complete hyperbolic manifold with one cusp, any symmetry of the Dehn filling restricts to a symmetry of the original manifold. This is the content of our next result. In fact the proof of this exceptional symmetry theorem follows the proof of the the hyperbolic Dehn surgery theorem. We refer to the exposition of the proof of the hyperbolic Dehn theorem given in, [3] in the following proof.

Theorem 5 (Exceptional Symmetry Theorem). *Let M be a complete hyperbolic 3-manifold with one cusp end. This implies it is diffeomorphic to the interior of a compact manifold with boundary a torus. Picking a basis for the first homology of this torus allows one to identify Dehn fillings with the extended rational numbers. For all but a finite number of $p/q \in \overline{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, the manifold $M(p/q)$ is hyperbolic and the isometry group $\text{Isom}(M(p/q))$ is isomorphic to a subgroup of $\text{Isom}(M)$.*

Proof. First notice that the hypothesis that M has one cusp end and is thus diffeomorphic to the interior of a compact manifold with boundary a union of tori implies that the complete hyperbolic metric on M has finite volume ([3] page 157). Pick a basis for the first homology of the boundary torus and use it to identify the collection of Dehn fillings with the extended rational numbers. Set

$$S := \{p + qi \in \mathbb{Z}[i] \mid \gcd(p, q) = 1, M(p/q) \text{ admits a hyperbolic structure}\}.$$

Thurston's Dehn surgery theorem (Theorem E.5.1 of [3]) implies that S contains all but a finite number of elements of $\mathbb{Z}[i]$ with $\gcd(p, q) = 1$. The idea of the proof is to show that all but a finite number of the manifolds corresponding to elements of S have a unique shortest geodesic with complement isomorphic to M . The result follows since isometries must take shortest geodesics to shortest geodesics.

Each of the manifolds corresponding to an element of S has finite volume since each is compact. The collection of all finite volume hyperbolic 3-manifolds may be topologized with the geometric topology ([3], chapter E). This collection will be denoted by \mathcal{F} .

Now recall the thick-thin decomposition of hyperbolic 3-manifolds. For $\epsilon > 0$, the thick part of a manifold N is the set of all points with injectivity radius greater than ϵ (denoted $N_{(\epsilon, \infty)}$) and the thin part is the set of all points with injectivity radius less than or equal to ϵ , and it is denoted by $N_{(0, \epsilon]}$. The Margulis lemma, [12], implies that there is a constant independent of the 3-manifold such that the ϵ -thin part of a finite volume, orientable 3-manifold will be a union of tubes $D^2 \times S^1$ about short closed geodesics and cusps $T^2 \times [0, \infty)$, provided ϵ is smaller than this Margulis constant, [3]. Since our starting manifold, M , has just one cusp end, the thin part will consist of the one cusp plus a finite collection of tubes. Taking the ϵ sufficiently small we can be sure that the thin part has no tubes.

The manifolds in a neighborhood of a fixed hyperbolic 3-manifold will be similar to it in a number of ways. In particular there is an $\epsilon > 0$ and neighborhood (in the geometric topology) of our starting manifold, M , such that the ϵ -thick part of every manifold in this neighborhood is homeomorphic to the ϵ -thick part of M . This follows from the proof of Theorem E.2.4 of [3]. Use \mathcal{U} to denote this neighborhood of M in \mathcal{F} .

The next step in our proof relies on the existence of a special ideal triangulation, Δ , of M . This triangulation exists by a theorem of Epstein and Penner [10]. Roughly speaking the hyperbolic structures on an ideal tetrahedron are parameterized by complex numbers with positive imaginary part. The collection of these complex numbers can be used to parameterize hyperbolic metrics on M . In order to be a metric the dihedral angles around each edge must sum to 2π . This condition may be expressed as a polynomial equation in these complex parameters. The space of solutions to these equations is denoted $\mathcal{H}_\Delta(M)$ and it corresponds to a collection of not necessarily complete hyperbolic metrics on M . See chapter E.6 of [3].

A countable collection of points in $\mathcal{H}_\Delta(M)$ correspond to manifolds having M or a closed hyperbolic manifold as a completion. Denote this set by \mathcal{H}' , and the element corresponding to M by z_∞ . Since each of these manifolds have finite volume, there is a natural map $\mathcal{H}' \rightarrow \mathcal{F}$ taking any solution to the compatibility equations to the corresponding completion. This map is continuous at z_∞ with respect to the topology on \mathcal{H}' as a subspace of Euclidean space, and the geometric topology on \mathcal{F} . See Proposition E.6.29 on page 266 of [3]. It follows that there is an open set about z_∞ that maps into \mathcal{U} . Let \mathcal{Def} be this open set. By Proposition E.6.23 on page 255 of [3], the image of the map from \mathcal{Def} to $S \cup \{\infty\}$ obtained by associating $p + qi$ to any $z \in \mathcal{Def}$ that maps to $M(p/q) \in \mathcal{U}$, contains a neighborhood of infinity. Thus it contains all but a finite number of the $p + qi$.

Given a $p + qi$ in the image of \mathcal{Def} corresponding to a closed manifold and an isometry, f , of the corresponding $M(p/q)$, we see that f must take the thin part to itself setwise since the points in the thin part are defined geometrically. It follows that f takes the complement of the closure of the thin part to itself. This is just the interior of the thick part. Since we took ϵ small enough that the thin part of M has only one component, we know that the thin part is just a product $T^2 \times \mathbb{R}$ so that the interior of the thick part is diffeomorphic to M . However, the original neighborhood, \mathcal{U} was chosen so that the thick part of each element of this neighborhood would be diffeomorphic to the thick part of M .

Thus we may associate to each isometry of $M(p/q)$ in this family the diffeomorphism (and hence mapping class) of M obtained by restricting f to the interior of the thick part of $M(p/q)$. One might worry that this could send a non-trivial isometry to a trivial mapping class, but that would be worrying too much. Indeed, the fundamental group of $M(p/q)$ is obtained by adding one relation to the fundamental group of M which we now identify with the interior of the thick part. One can take (representatives of) generators of the fundamental group of M and see what happens to each under the image of f . Assuming that $f|_M$ is isotopic to the identity, one can let $f|_{M,t}$ represent the isotopy. This gives natural tails for each map in the family of the isotopy. Indeed, when x_0 is the base point $\gamma_t(s) = f|_{M,st}(x_0)$ is the natural tail. The automorphism induced by f takes a generator α to $\gamma_1 * (f \circ \alpha) * \gamma_1^{-1}$. This is homotopic (rel x_0) to α via $\gamma_t * (f|_{M,t} \circ \alpha) * \gamma_t^{-1}$, so the induced automorphism is trivial implying that f itself was trivial to begin with. \square

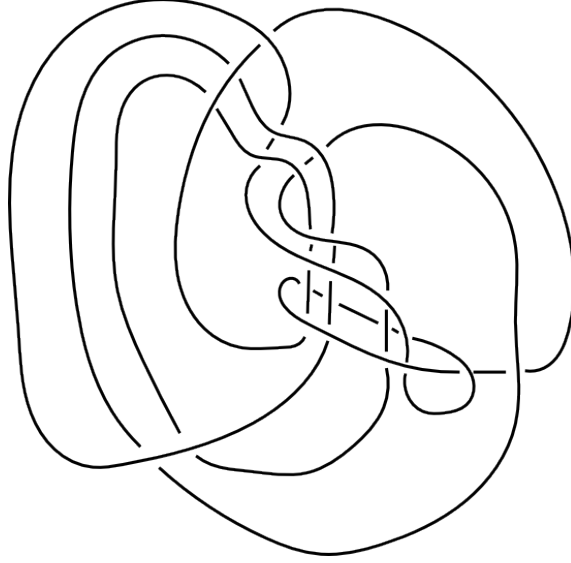


Figure 10: An asymmetric knot with an exceptional symmetry

The knot in figure 10 demonstrates the exceptional symmetries that can occur as described in the theorem. It has no symmetries, so all but a finite number of Dehn fillings of this knot have no symmetries. This implies that none of these infinitely many manifolds 2-fold branched cover any 3-manifold. However, -2 -surgery on this large knot is equivalent to 2-surgery on the 8_6

knot. This knot has the dihedral group of order 4 as a symmetry group, and these symmetries extend to the surgeries on the knot. Two of the involutions fix points in the knot so the corresponding quotients are S^3 . The axis of the other involution avoids the knot. Even so, the knot projects to an unknot in the quotient. The induced framing is 1 so the quotient is still S^3 . To see that -2 surgery on the large knot is the same as 2 surgery on the 8_6 knot, one starts with the 9^2_{35} link displayed in figure 11. Each component of this link is unknotted. Blowing down the 1-framed component leads to the large knot from figure 10. Blowing down the -1 -framed component leads to the 8_6 knot.

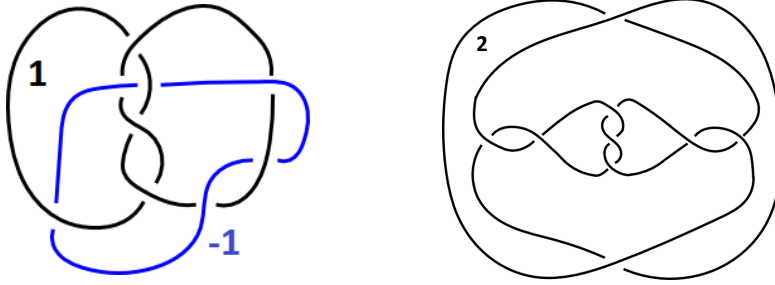


Figure 11: The 9^2_{35} link and 8_6 knot

The theorem states that every symmetry of a generic filling is a symmetry of the open manifold. It does not state that every symmetry of the open manifold extends to the filling. We have seen this in the example of the 8_{17} knot displayed in figure 6. In fact, for homological reasons a symmetry of type (S^0, S^0) can only extend across trivial or zero surgery. On the other hand we have seen that every symmetry of type (S^1, S^0) , or (S^1, \emptyset) does extend across every surgery, so the symmetries extend for the examples that we care most about.

One of the reasons the 5_2 knot from our first example is a good knot to consider is that it also admits exceptional symmetries, as well as exceptional surgeries. It has symmetry group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with two involutions of type (S^1, S^0) and one involution of type (S^1, \emptyset) . By our earlier discussion these symmetries extend across all surgeries on this knot, and for all but a finite number of exceptions these surgered manifolds have symmetry $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The manifold $S^3_{5_2}(1/3)$ is hyperbolic and has symmetry $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as expected.

The manifold $S^3_{5_2}(1)$ is not hyperbolic. To see this we do the same trick

that we did to understand the exceptional symmetry in the large knot – we find a two component link half way in between. A Rolfsen twist on an unknot that links the middle $3/2$ twisted band in figure 5 will untwist it two times. The framing on this new curve will be $1/2$. This operation will unknot the 5_2 knot. Since the framing on the new unknot is 1, we can blow it down to see that this manifold is the same as $1/2$ surgery on the right hand trefoil. This in turn is the Seifert Fiber Manifold over S^2 with three singular fibers and invariants $\{1, (Oo, 0), (-2, 1), (-3, 1), (-11, 2)\}$.

The manifold $S^3_{5_2}(1/2)$ has symmetry $D_{2.4}$ not $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as expected. To see this, note that a Rolfsen twist about an unknot that links one of the half-twisted bands and reverses the crossing will unknot the 5_2 knot resulting in a link of two unknotted and algebraically unlinked components. One will have filling $1/2$ and the other will have filling 1. Doing the Rolfsen twist to untwist the $1/2$ -framed component will result in 1 filling on the 8_3 knot. The $D_{2.4}$ is then manifest.

Branched Virtual Fibration

The virtual fibration theorem is a major new result in the theory of 3-manifolds. It states that any hyperbolic 3-manifold admits a finite cover that fibers over the circle. Thurston asked it as a question in 1982. In 2007 Agol proved that any manifold with what is known as a virtually residually finite rationally solvable fundamental group is virtually fibered. This year Wise and Agol proved that the fundamental group of any closed hyperbolic 3-manifold has this property, [2, 32].

If finite cover is relaxed to finite *branched* cover one can see that any 3-manifold has a 2-fold branched cover that fibers over the circle. We now prove this branched virtual fibration result. The proof begins with the well-known fact that any 3-manifold has an open book decomposition. There are many ways to see that 3-manifolds admit such open book decompositions. One way starts with a representation of the given manifold M as an irregular 3-fold branched cover branched over a braid. Given such a cover express the 3-sphere as a union $(S^1 \times D^2) \cup (D^2 \times S^1)$ with the braid in the second factor $D^2 \times S^1$ such that the projection to the S^1 factor restricted to the braid has no critical points. The open book representation is constructed by composing the branched covering projection from the manifold with the following map

from the 3-sphere to the disk D^2 :

$$f : S^3 = \{(z, w) \in \mathbb{C}^2 \mid \max(|z|, |w|) = 1\} \rightarrow D^2; f(z, w) = w.$$

The inverse image of the braid axis $S^1 \times \{0\} \subseteq S^1 \times D^2$ is called the binding. It is easy to see that the complement of the binding fibers over the circle. In fact the map $M \rightarrow D^2$ takes the complement of the binding to $D^2 - \{0\}$ and the fibration in question is obtained by mapping these points to S^1 by $v \mapsto |v|^{-1}v$.

Theorem 6 (Branched Virtual Fiberings). *Every closed orientable 3-manifold has a 2-fold branched cover that fibers over a circle.*

Proof. First notice that the involution $\tau : S^1 \times [-1, 1] \rightarrow S^1 \times [-1, 1]$ given by $\tau(z, t) = (-z, -t)$ induces a 2-fold branched cover to the quotient under the τ action, and this quotient is a disk. Denote this branched covering by $p : S^1 \times [-1, 1] \rightarrow D^2$. Let $p_1 : S^1 \times [-1, 1] \rightarrow S^1$ be projection onto the first factor. Given an open book presentation of a closed orientable 3-manifold, $b : M \rightarrow D^2$, the desired 2-fold branched cover of M is just the pull-back of M by the 2-fold branched cover of the disk, and the fiber bundle projection to the circle is just the composition of the induced map to $S^1 \times [-1, 1]$ with projection onto the first factor. The pull-back is defined by

$$p^*M := \{(z, t, x) \in S^1 \times [-1, 1] \times M \mid p(z, t) = b(x)\}.$$

This argument is summarized in the following commutative diagram:

$$\begin{array}{ccc} p^*M & \longrightarrow & S^1 \times [-1, 1] \xrightarrow{p_1} S^1 \\ \downarrow & & \downarrow p \\ M & \xrightarrow{b} & D^2. \end{array}$$

Basically, this replaces a copy of the product of the binding and D^2 with a copy of the product of the binding with an annulus and duplicates the fibered portion of M . This is depicted in the following figure.

□

Quotient Tabulation

The symmetries of knots with fewer than 11 crossings were tabulated in [14, 20]. For many knots this is all the information that is needed to understand

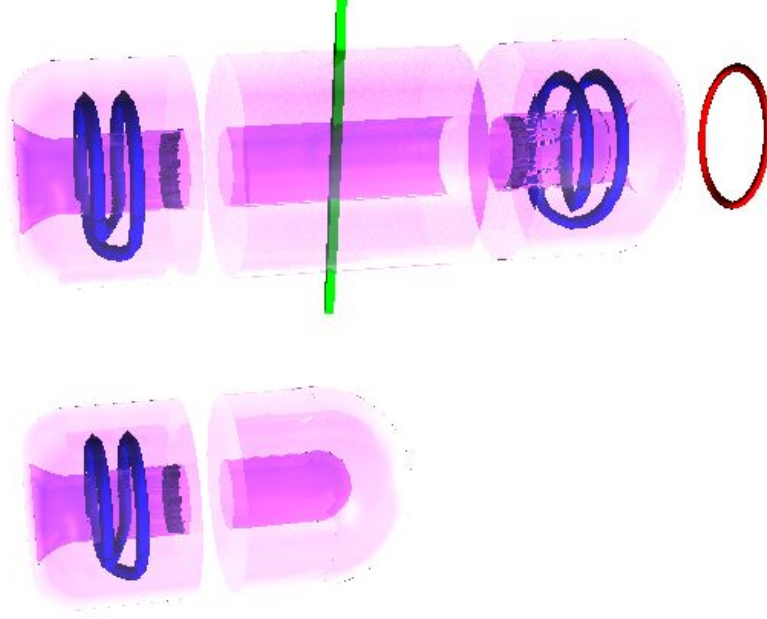


Figure 12: Branched virtual fibering

the possible quotients by 2-fold branched covers. However, it is not always possible to infer the type of the symmetry of an element of one of these groups from the tabulated information. Furthermore, for symmetries of type (S^1, \emptyset) one needs to know if the image of the knot in the quotient is knotted or unknotted in order to decide if the homology sphere fillings 2-fold branched cover a non-trivial manifold. Rather than just including the knots that need new information (knotted or unknotted quotient) and referring the reader to the earlier tabulations for the rest, we include all knots in this tabulation. We do just concentrate on the involutions, but remark here that the following knots have higher order symmetries: 4_1 , 6_3 , 7_4 , 7_7 , 8_3 , 8_9 , 8_{12} , 9_{10} , 9_{17} , 9_{23} , 9_{31} , 10_{17} , 10_{33} , 10_{37} , 10_{43} , 10_{45} , 10_{157} (all with D_4 symmetry), 9_{41} , 9_{47} , 9_{49} (all with D_3 symmetry), 9_{35} , 9_{40} , 9_{48} , 10_{75} (all with D_6 symmetry), 8_{18} (D_8 symmetry), and 10_{123} (D_{10} symmetry).

Census

Symmetry Type	Knot List
No symmetry – generic surgeries on these knots do not 2-fold branched cover any manifold.	$9_{32}, 9_{33}, 10_{80}, 10_{82}, 10_{83},$ $10_{84}, 10_{85}, 10_{86}, 10_{87}, 10_{90},$ $10_{91}, 10_{92}, 10_{93}, 10_{94}, 10_{95},$ $10_{102}, 10_{106}, 10_{107}, 10_{110},$ $10_{117}, 10_{119}, 10_{148}, 10_{149},$ $10_{150}, 10_{151}, 10_{153}$
Only type (S^0, S^0) symmetry – generic surgeries on these knots do not 2-fold branched cover any manifold, but 0 surgery will be a 2-fold cover of a non-orientable manifold.	$8_{17}, 10_{79}, 10_{81}, 10_{88}, 10_{109},$ $10_{115}, 10_{118}$
Only type (S^1, S^0) symmetry – generic surgeries on these knots will be 2-fold branched covers over S^3 , but do not 2-fold branched cover any other manifold.	$8_{10}, 8_{16}, 8_{20}, 9_{22}, 9_{24}, 9_{25},$ $9_{29}, 9_{30}, 9_{34}, 9_{36}, 9_{38}, 9_{39},$ $9_{41}, 9_{42}, 9_{43}, 9_{44}, 9_{45}, 9_{47},$ $9_{49}, 10_{46}, 10_{47}, 10_{48}, 10_{49},$ $10_{50}, 10_{51}, 10_{52}, 10_{53}, 10_{54},$ $10_{55}, 10_{56}, 10_{57}, 10_{59}, 10_{62},$ $10_{65}, 10_{70}, 10_{71}, 10_{72}, 10_{73},$ $10_{77}, 10_{89}, 10_{96}, 10_{97}, 10_{100},$ $10_{101}, 10_{103}, 10_{104}, 10_{105},$ $10_{108}, 10_{111}, 10_{112}, 10_{113},$ $10_{114}, 10_{116}, 10_{121}, 10_{125},$ $10_{126}, 10_{127}, 10_{128}, 10_{129},$ $10_{130}, 10_{131}, 10_{132}, 10_{133},$ $10_{134}, 10_{135}, 10_{137}, 10_{140},$ $10_{143}, 10_{152}, 10_{154}, 10_{156},$ $10_{158}, 10_{159}, 10_{160}, 10_{161},$ $10_{162}, 10_{163}, 10_{164}, 10_{165}$

Symmetry Type	Knot List
Only type (S^1, \emptyset) symmetry with unknotted quotient – generic surgeries on these knots will be 2-fold branched covers over a lens space. This unique possible quotient will be S^3 exactly when the filling yields an integral homology sphere upstairs, i.e. $1/n$ filling.	$10_{67}, 10_{147}$
Only type (S^1, \emptyset) symmetry with knotted quotient – generic surgeries on these knots will be 2-fold branched covers over some non-simply-connected manifold, but do not 2-fold branched cover S^3 .	10_{98}
Both types (S^1, S^0) symmetry and (S^1, \emptyset) symmetry with unknotted quotient – generic surgeries on these knots will be 2-fold branched covers over S^3 , as well as some lens space (unless the filling yields an integer homology sphere in which case the only possible quotient will be S^3).	$4_1, 5_2, 6_1, 6_2, 6_3, 7_2, 7_3, 7_4,$ $7_5, 7_6, 7_7, 8_1, 8_2, 8_3, 8_4, 8_6,$ $8_7,$ $9_2, 9_3, 9_4, 9_5, 9_6, 9_7, 9_8, 9_9,$ $9_{10}, 9_{11}, 9_{12}, 9_{13}, 9_{14}, 9_{15},$ $9_{17}, 9_{18}, 9_{19}, 9_{20}, 9_{21}, 9_{23},$ $9_{26}, 9_{27}, 9_{31}, 9_{35}, 9_{37}, 9_{46},$ $9_{48},$ $10_1, 10_2, 10_3, 10_4, 10_5, 10_6,$ $10_7, 10_8, 10_9, 10_{10}, 10_{11},$ $10_{12}, 10_{13}, 10_{14}, 10_{15}, 10_{16},$ $10_{17}, 10_{18}, 10_{19}, 10_{20}, 10_{21},$ $10_{22}, 10_{23}, 10_{24}, 10_{25}, 10_{26},$ $10_{27}, 10_{28}, 10_{29}, 10_{30}, 10_{31},$ $10_{32}, 10_{33}, 10_{34}, 10_{35}, 10_{36},$ $10_{37}, 10_{38}, 10_{39}, 10_{40}, 10_{41},$ $10_{42}, 10_{43}, 10_{44}, 10_{45}, 10_{68},$ $10_{69}, 10_{74}, 10_{75}, 10_{145}, 10_{146}$
Both types (S^1, S^0) symmetry and (S^0, \emptyset) symmetry – generic surgeries on these knots will only be 2-fold branched covers over S^3 .	$10_{99}, 10_{123}$

Symmetry Type	Knot List
Both types (S^1, S^0) symmetry and (\emptyset, \emptyset) symmetry – generic surgeries on these knots will be 2-fold branched covers over S^3 , as well as a non-trivial 3-manifold.	$10_{155}, 10_{157}$
Both types (S^1, S^0) symmetry and (S^1, \emptyset) symmetry with knotted quotient – non-trivial surgeries on these knots will be 2-fold branched covers over S^3 , as well as some non-trivial 3-manifold.	$8_5, 8_{15}, 8_{21}, 9_{16}, 9_{28}, 9_{40}, 10_{58}, 10_{60}, 10_{61}, 10_{63}, 10_{66}, 10_{76}, 10_{78}, 10_{120}, 10_{122}, 10_{136}, 10_{138}, 10_{139}, 10_{141}, 10_{142}, 10_{144}$
Torus knots with both types (S^1, S^0) symmetry and (S^1, \emptyset) symmetry with unknotted quotient – any surgery on one of these knots will be 2-fold branched covers over S^3 , as well as some lens space (unless the filling yields an integer homology sphere in which case the only possible quotient will be S^3).	$3_1, 5_1, 7_1, 9_1$
Torus knots with both types (S^1, S^0) symmetry and (\emptyset, \emptyset) symmetry – any surgery on one will be a 2-fold branched cover over S^3 , as well as a 2-fold cover (or branched cover) over some other manifold.	10_{124}

References

- [1] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and stringy topology*, volume 171 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2007.
- [2] I. Agol. The virtual haken conjecture, with an appendix by i. agol, d. groves, j. manning. *Preprint*, 2012.

- [3] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.
- [4] Laurent Bessières, Gérard Besson, Sylvain Maillot, Michel Boileau, and Joan Porti. *Geometrisation of 3-manifolds*, volume 13 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2010.
- [5] Michel Boileau, Bernhard Leeb, and Joan Porti. Geometrization of 3-dimensional orbifolds. *Ann. of Math. (2)*, 162(1):195–290, 2005.
- [6] Michel Boileau, Sylvain Maillot, and Joan Porti. *Three-dimensional orbifolds and their geometric structures*, volume 15 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2003.
- [7] Francis Bonahon and L. C. Siebenmann. *New Geometric Splittings of Classical Knots and the Classification and Symmetries of Arborescent Knots*, volume XX of *Progress in Mathematics*. Geometry and Topology Monographs, xxx.
- [8] Glen E. Bredon. *Introduction to compact transformation groups*. Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 46.
- [9] Huai-Dong Cao and Bennett Chow. Recent developments on the Ricci flow. *Bull. Amer. Math. Soc. (N.S.)*, 36(1):59–74, 1999.
- [10] D. B. A. Epstein and R. C. Penner. Euclidean decompositions of non-compact hyperbolic manifolds. *J. Differential Geom.*, 27(1):67–80, 1988.
- [11] Ralph H. Fox. A note on branched cyclic covering of spheres. *Rev. Mat. Hisp.-Amer. (4)*, 32:158–166, 1972.
- [12] M. Gromov. Almost flat manifolds. *J. Differential Geom.*, 13(2):231–241, 1978.
- [13] John Hempel. *3-manifolds*. AMS Chelsea Publishing, Providence, RI, 2004. Reprint of the 1976 original.
- [14] Shawn R. Henry and Jeffrey R. Weeks. Symmetry groups of hyperbolic knots and links. *J. Knot Theory Ramifications*, 1(2):185–201, 1992.

- [15] Hugh M. Hilden. Three-fold branched coverings of S^3 . *Amer. J. Math.*, 98(4):989–997, 1976.
- [16] Ulrich Hirsch. Über offene Abbildungen auf die 3-Sphäre. *Math. Z.*, 140:203–230, 1974.
- [17] Craig D. Hodgson and Jeffrey R. Weeks. Symmetries, isometries and length spectra of closed hyperbolic three-manifolds. *Experiment. Math.*, 3(4):261–274, 1994.
- [18] Michael Kapovich. *Hyperbolic manifolds and discrete groups*, volume 183 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2001.
- [19] Bruce Kleiner and John Lott. Notes on Perelman’s papers. *Geom. Topol.*, 12(5):2587–2855, 2008.
- [20] Kouzi Kodama and Makoto Sakuma. Symmetry groups of prime knots up to 10 crossings. In *Knots 90 (Osaka, 1990)*, pages 323–340. de Gruyter, Berlin, 1992.
- [21] P. B. Kronheimer and T. S. Mrowka. Witten’s conjecture and property P. *Geom. Topol.*, 8:295–310 (electronic), 2004.
- [22] José M. Montesinos. Three-manifolds as 3-fold branched covers of S^3 . *Quart. J. Math. Oxford Ser. (2)*, 27(105):85–94, 1976.
- [23] John W. Morgan and Hyman Bass, editors. *The Smith conjecture*, volume 112 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1984. Papers presented at the symposium held at Columbia University, New York, 1979.
- [24] John W. Morgan and Frederick Tsz-Ho Fong. *Ricci flow and geometrization of 3-manifolds*, volume 53 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2010.
- [25] G. D. Mostow. Quasi-conformal mappings in n -space and the rigidity of hyperbolic space forms. *Inst. Hautes Études Sci. Publ. Math.*, (34):53–104, 1968.
- [26] Grisha Perelman. The entropy formula for the ricci flow and its geometric applications. <http://arxiv.org/abs/math.DG/0211159>.

- [27] Grisha Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds. <http://arxiv.org/abs/math.DG/0307245>.
- [28] Grisha Perelman. Ricci flow with surgery on three-manifolds. <http://arxiv.org/abs/math.DG/0303109>.
- [29] Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series*. Publish or Perish Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.
- [30] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.
- [31] William P. Thurston. Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds. *Ann. of Math. (2)*, 124(2):203–246, 1986.
- [32] D. Wise. The structure of groups with a quasi-convex hierarchy. *Preprint*, 2012.